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Title of the talk

On lower density operators

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Abstract

Let (X, τ_0) be a topological T_1 space and $\mathcal{B}$ - Borel σ-algebra on (X, τ_0). Let $\mathcal{P}(X)$ denote the power set of X. Fix a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ containing all singletons such that $\mathcal{I} \cap \tau_0 = \{\emptyset\}$. By $\mathcal{S}$ we will denote the σ-algebra generated by $\mathcal{B} \cup \mathcal{I}$. In this way we obtain a measurable space $(X, \mathcal{S}, \mathcal{I})$.

Let $\Phi : \mathcal{S} \to 2^X$ be a *lower density operator* on $\mathcal{S}$, i.e. for each $A, B \in \mathcal{S}$, the following conditions are fulfilled:

1. $\Phi(\emptyset) = \emptyset, \quad \Phi(X) = X$;

2. if $A \triangle B \in \mathcal{I}$, then $\Phi(A) = \Phi(B)$;

3. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;

4. $\Phi(A) \bigtriangleup A \in \mathcal{I}$,

where $A \Delta B$ is a symmetric difference of A and B.

If the pair $(\mathcal{S}, \mathcal{I})$ fulfills C.C.C. or for each $A \subset X$ there exists a measurable hull of A then the family

$$\tau_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\} = \{\Phi(A) \setminus I : A \in \mathcal{S}, I \in \mathcal{I}\}$$

forms a topology on X (compare [1], [2]), called an abstract density topology generated by Φ.

By $\mathcal{L}$ we denote family of all Lebesgue measurable sets. Symbols $\mathrm{Int}(A)$ and $\overline{A}$ stand for the interior and closure of A with respect to τ_0.

It is well known that if τ_0 is the Euclidean topology on the real line and Φ_d is the classical density operator then τ_{Φ_d} is the density topology τ_d (see [3]) and

$$\mathrm{Int}(A) \subset \Phi_d(A) \subset \overline{A} \tag{1}$$

for arbitrary $A \in \mathcal{L}$, so $\tau_0 \subset \tau_d$.

It seems to be interesting that for arbitrary lower density operator defined on $\mathcal{L}$ the inclusions analogous to (1) are not necessary.

We consider two examples of lower density operators, for which the inclusions from (1) are not fulfilled. For this purpose the measure-preserving bijections on the real line are applied.

We also compare the operator Φ_f generated by measure-preserving bijection f with classical density operator Φ_d and prove that the symmetric difference $\Phi_f(A) \bigtriangleup \Phi_d(A)$ can be an uncountable set or set of the second category for some measurable A.

References

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[2] J. Lukeš, J. Malý, L. Zajiček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Mathematics 1189, Springer-Verlag, 1986.

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